INSTABILITY IN THE HIGHER MODES OF A NONLINEAR EQUATION

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The linear approximation is used to study the stability of two- and three-dimensional higher-order modes of a nonlinear wave equation against exponentially increasing perturbations. For all the nonlinear models considered the higher modes are unstable; the number of exponentially increasing perturbations and their growth rate are determined by the mode number and the form of the nonlinear relationship. Numerical tests are described in the parabolic approximation on the stability of the first axially symmetric mode against small amplitude perturbations and the conditions are determined under which higher-order modes can be observed.

In the simplest model of a noninertial nonlinear medium the propagation of electromagnetic waves in the scalar parabolic approximation is described by the equation [1, 2].

$$i\frac{\partial\psi}{\partial\tau}+\Delta\psi+f(|\psi|^2)\psi=0, \tag{1}$$

where ψ is the complex envelope of the electromagnetic wave field, and $f(|\psi|^2)$ describes the nonlinear part of the dielectric permittivity. The possible $f(|\psi|^2)$ for which there exist finite stationary solutions of the type

$$\psi_n=\varphi_n(r)\exp(i\gamma\tau),\ n=0,\ 1,\ 2,\ \ldots,\ \gamma>0, \tag{2}$$

have been studied in [3-5] where rather general theorems on the existence of solutions (2) have been obtained. If the stationary field distributions (2) are to be observed experimentally, it is necessary that they be stable against small phase and amplitude perturbations. A detailed study of stability for Eq. (1) has only been made for the principal mode, which has constant sign [4, 6-8]. The stability of the higher alternating mode $\varphi_n(n\geqslant 1)$ has not been investigated.

Several propositions about the stability of the higher-order modes can be made from the results in [8-10]. Thus, for the stability of a stationary solution ψ_n it is sufficient that the solution should minimize the functional

$$H(\psi)=\int[|\nabla\psi|^2+\gamma|\psi|^2-F(|\psi|^2)]\,dv,\ F(|\psi|^2)=\int_0^{|\psi|^2}f(\eta)\,d\eta \tag{3}$$

in some subspace of the total Hilbert function space. For some functions $f(|\psi|^2)$ the principal mode ψ_0 minimizes the functional (3) and therefore in the corresponding nonlinear media ψ_0 is a stable solution [8]. The higher-order modes for arbitrary do not minimize (3) [8, 10] and this suggests that they are unstable.

In order to test this assumption we have studied the stability of the higher-order modes for various nonlinear models. In the linear approximation the existence of exponentially increasing perturbations of the form $(u+iv)\exp(i\gamma\tau+\Omega\tau)$ for a stationary solution ψ_n is determined by the existence of real eigenvalues of the system of equations

$$\Omega u=L_{0n}v;\ \Omega v=-L_{1n}u, \tag{4}$$

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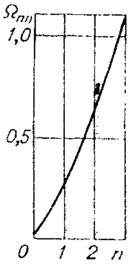

Fig. 1

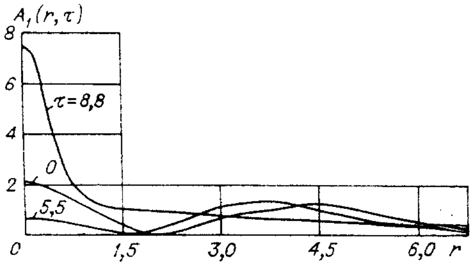

Fig. 2

where $L_{0n} = -\Delta + \gamma - f(\varphi_n^2)$; $L_{1n} = L_{0n} - 2\frac{df}{d\varphi_n^2}\varphi_n^2$. If the system (4) has a solution u, v, Ω, then ±u, ±v, -Ω, are also solutions. In what follows we limit the discussion to positive values of Ω. Since for n ⩾1 the variational method is inapplicable to (4) we have employed numerical integration of these equations on a BÉSM-4 computer.

The solution of (4) was sought for in the form

$$u_m = \overset{*}{u}_m(r) \times \begin{cases}\cos m\Phi \\ \sin m\Phi\end{cases}; \quad v_m = \overset{*}{v}_m(r) \times \begin{cases}\cos m\Phi \\ \sin m\Phi\end{cases}$$

for a two-dimensional Laplacian Δ and in the form

$$u_{lm} = \overset{*}{u}_{lm}(r)\,Y(\theta,\Phi); \quad v_{lm} = \overset{*}{v}_{lm}(r)\,Y_{lm}(\theta,\Phi)$$

for a three-dimensional Laplacian.

A knowledge of the asymptotic representation of $\overset{*}{u}_m(r)$, $\overset{*}{v}_m(r)$ [or $\overset{*}{u}_{lm}(r)$, $\overset{*}{v}_{lm}(r)$] for sufficiently large r enabled us to integrate (4) and to test that the boundary condition $\overset{*}{u}_m(r), \overset{*}{v}_m(r) \sim r^m$ (or $\overset{*}{u}_{lm}(r), \overset{*}{v}_{lm}(r) \sim r^l$) as $r \to 0$ was satisfied. If this condition was not or satisfied a correction was made to the eigenvalue and the calculations were repeated [11].

The principal result from the numerical analysis of (4) was that for all nonlinear models we considered ($f=\varphi^{2p}$, $0<p\leqslant 2$; $f=\varphi^2/(1+\varphi^2)^k$, $0\leqslant k\leqslant 1$) in both the two- and three-dimensional cases real eigenvalues of (4) were found for the higher modes n ⩾1. The number of exponentially increasing perturbations is determined by the mode number n, the values of γ, and the form of the nonlinear relationship $f(\varphi^2)$. With $f(\varphi^2)$ functions for which the principal mode ψ_0 is stable against exponentially increasing perturbations the number of nondegenerate eigenvalues of the system Ω_{nm} (or Ω_{nml}) for the mode ψ_n is not less than n and for a fixed mode number n the maximum eigenvalue Ω_n^* corresponds to the maximum possible m (or l). In this case all exponentially increasing perturbations depend on the angles Φ or Φ, θ. If ψ_0 is unstable against exponentially increasing perturbations, then the number of nondegenerate positive eigenvalues Ω_{nm} (or Ω_{nml}) for the mode ψ_n is not less than n + 1 and the maximum eigenvalue Ω_n^* corresponds to m = 0 (or l = 0). In this case there exist perturbations which do not depend on the angles.

If $f(\varphi^2)$ increases without bound as $\varphi^2\to\infty$, then the value of Ω_n^* also increases without bound as the mode number n increases. Figure 1 shows how Ω_{nm} varies with the mode number for the axially symmetric modes of a cubic medium $f=\varphi^2$, $\gamma=1$. If the maximum of the expression $|(df/d\varphi^2)\varphi^2|$ does not exceed M for $0\leqslant\varphi^2<\infty$, then M, as $|\Omega_n^*|\leqslant M$. From (4) we can get

$$\Omega(\langle u|u\rangle + \langle v|v\rangle) = \langle u|L_{0n}|v\rangle - \langle v|L_{1n}|u\rangle = 2\left\langle v\left|\frac{df}{d\varphi_n^2}\varphi_n^2\right|u\right\rangle,$$

and thus

$$|\Omega|(\langle u|u\rangle + \langle v|v\rangle) \leqslant \max\left|\frac{df}{d\varphi_n^2}\varphi_n^2\right|(\langle u|u\rangle + \langle v|v\rangle)$$

and

$$|\Omega_n^*| \leqslant \max\left|\frac{df}{d\varphi_n^2}\varphi_n^2\right| = M.$$

If a medium with a saturating nonlinearity is approximated by the function $f=\varphi^2/1+\varphi^2$, then for all higher modes $|\Omega_n^*|\leqslant 0.25$. This result has been confirmed by numerical solution of (4).

Thus, if the principal mode is unstable the higher modes cannot be observed because there are exponentially increasing perturbations which both depend on and do not depend on the angles. If the principal mode is stable, then the question of the stability of higher modes which are independent of angle is an open one. According to linear stability theory there are always perturbations for the solutions (2) [from Eq. (1)] which increase linearly with the time τ [7-8]. Among these perturbations are some which do not depend on the angles. The sufficient condition for the limited growth of these perturbations in the higher modes is not satisfied. Thus, even when the principal mode is stable the higher modes can be unstable against perturbations which are independent of the angles.

In order to study this case we have carried out numerical experiments to analyze the growth of small perturbations of the form $\delta\psi=a\exp(-r^2/l^2)$ for the first axially symmetric mode ψ_1 with $f=|\psi|^2/(1+|\psi|^2)^k$, $0<k\leqslant 1$. In this case ψ_0 is stable against exponentially increasing perturbations and all such perturbations for the higher modes depend on the angle Φ. We might note that for the $f(|\psi|^2)$ relationships we chose, the propagation of a light beam is not accompanied by singularities. This follows from an analysis in the paraxial approximation and also from the fact that in this case infinite energy is required for the formation of a singularity. Thus, if there is an instability it must be oscillatory. This conclusion is confirmed by the numerical solutions of (1) with the boundary conditions.

$$\frac{\partial\psi}{\partial r}(0,\tau)=0,\quad \psi(\infty,\tau)=0.$$

Equation (1) has been approximated by a second-order implicit three-layer finite-difference scheme in both variables [12]. A regular model was used for the radial variable r. After correct transfer of the boundary conditon from infinity [13] the resultant system of algebraic equations was solved by the pivotal method. The accuracy of the calculations was monitored from the conservation of the invariants

$$I^{(1)}=\int|\varphi|^2dv;$$
$$I^{(2)}=\int[|\nabla\psi|^2-F(|\psi|^2)]dv.$$

The worst relative accuracy of conservation for the results quoted was $I^{(1)}\sim 0.5\cdot 10^{-4}$, $I^{(2)}\sim 3\cdot 10^{-4}$.

An analysis of the radial distribution of the field amplitude in the propagation of the perturbed first mode showed the presence of considerable oscillations in the radial distribution and related oscillations in the amplitude at the center of the beam. In Fig. 2 we show the radial distribution of the field amplitude of the first mode

$$|\psi_1(r,\tau)|=A_1(r,\tau) \text{ for } f=|\varphi|^2/(1+|\varphi|^2)^{1/4};\ \gamma=0.5;\ a=-0.4;$$

l = 1 for several values of τ. It can be seen that the field energy varies from being concentrated at the center of the beam to being concentrated at the periphery. In the first case the amplitude distribution is close to that in the principal mode, whose energy is approximately equal to that of the first mode. We can thus give an upper limit to the maximum possible oscillation amplitude

$$\frac{A^*(0,\tau)}{A(0,0)}\sim\left(\frac{I_1}{I_0}\right)^{\frac{1-k}{2k}};\quad f\sim|\psi|^{2(1-k)},\ |\psi|\to\infty,$$

where I_0 and I_1 are the energies of the principal and first modes for the given nonlinear relationships. Thus as $k\to 1$ (saturation of the nonlinearity) the oscillation amplitude must decrease. Figure 3 shows how the amplitude of the first mode $A_1(0,\tau)$ varies with τ for γ = 0.5 at values of k = 0.25, 0.375, 0.5, 0.75 (curves 1-4, respectively) and at a fixed initial perturbation a = -0.4, l = 1. The reduction in oscillation amplitude with increase in k is obvious from the graphs. Usually $I_1/I_0\gg 1$, and so for small k the ratio $A^*(0,\tau)/A(0,0)$ can reach high values. The relatively large error in our calculations means that it was not

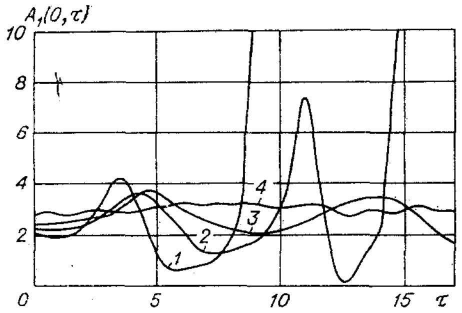

Fig. 3

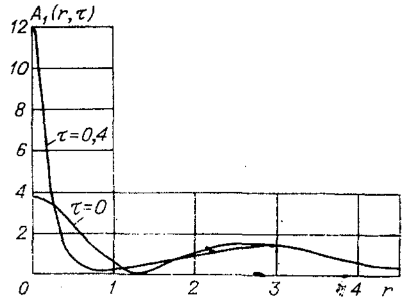

Fig. 4

always possible to determine this ratio and to continue the calculations until the maximum field amplitude $A^*(0, \tau)$ was reached.

If $f=|\psi|^{2p}$, $p \geqslant 1$, singularities can be formed during the propagation of a perturbed mode [2, 14] and the nature of the breakdown in the higher modes is different from the case where no singularities are formed. An analysis of the radial distribution of amplitude of the perturbed first mode with axial symmetry when $p \geqslant 1$ has shown that the singularity is formed mainly as a result of the field in the central part of the beam and the field distribution at the periphery changes very little during the propagation process. This is due to the fact that when p = 1 an energy $I_0 \ll I_1$, is focused into a singularity, but when p > 1 the focused energy is infinitesimally small. There are no noticeable oscillations in the radial amplitude distribution. Figure 4 shows the radial amplitude distributions of the first mode $A_1(r, \tau)$ for $f=|\psi|^2$; $\gamma=1$; $a=0{,}4$; $l=1$.

Thus from a mathematical point of view the higher modes in all the models of a nonlinear medium which we have considered turn out to be unstable. Nevertheless, higher modes in a medium with a saturating nonlinearity can be observed if the initial field distribution does not depend on the angles and is close to the initial distribution of the corresponding modes.

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